N-DIMENSIONAL DUST STATIC SPHERICALLY SYMMETRIC NON-VACUUM SOLUTIONS IN $f(R)$ THEORY OF GRAVITY

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Abstract
In the present paper, we have investigated *N*-dimensional static spherically symmetric non-vacuum solution of the field equation in metric gravity using dust matter and constant Ricci scalar curvature. The density of dust matter and the function of the Ricci scalar curvature will also be evaluated at constant scalar curvature.
Keywords: *f(R)* theory of gravity, spherically symmetric solutions, dust solution, non-vacuum solution.

1. Introduction

The study of the solutions in f (R) theory of gravity is an important source of inspiration for all researchers in the field of general theory of relativity. Many authors have shown keen interest in exploring different issues in f (R) theories of gravity. The f (R) gravity theory is modified by replacing R with f (R) in the standard Einstein's Hilbert action and f (R) is a general function of the Ricci scalar. If we consider R in place of f (R) then the action of standard Einstein's Hilbert can be obtained. In the f (R) theory of gravity there are two approaches to find out the solutions of modified Einstein's field equations. The first approach is called metric approach and second one is known as Palatini formalism.

S. N. Pandey (2008) has developed a higher order theory of gravitation based on a Lagrangian density consisting of a polynomial of scalar curvature to obtain gravitational wave equations conformally flat. We observe that the many researchers have obtained the vacuum solutions of the field equations in metric f (R) gravity using the concept of spherical symmetry. The static spherically symmetric vacuum solutions of the field equation in f (R) theory of gravity have been obtained by Multamaki and Vilja (2006), Carames and Bezeera (2009) have obtained spherically symmetric vacuum solutions in higher dimension. Using Noether symmetry Capozziello et. al. (2007) have investigated spherically symmetric solutions. Sharif and Kausar (2011) have studied non vacuum static spherically symmetric solutions of the field equations in f (R) theory of gravity in the presence of dust fluid in four dimensional space-time. In recent years superstring and other field theories provoked great interest among theoretical physicists in studying physics of higher dimension.

Thus in the present paper we propose to solve N-dimensional field equations in f (R) theory of gravity using metric approach with constant scalar curvature and obtain non-vacuum static spherically symmetric solutions in the presence of dust fluid. The density ρ of dust matter and the Ricci scalar curvature function f (R) will also be evaluated at constant scalar curvature $R = R_0$.The corresponding field equations in f (R) theory of gravity in V_n are given by:

$$F(R)R_{ij}-\frac{1}{2}f(R)g_{ij}-\nabla_i\nabla_j\,F(R)+g_{ij}\,\Box F(R)=kT_{ij}\,,\quad (i,j=1,2,3,\cdots,n) \tag{1}$$

where $F(R)\equiv\dfrac{df(R)}{dR}$, $\Box\equiv\nabla^i\,\nabla_i$, ∇_i is the covariant derivative , $k(=8\pi)$ is the coupling constant in gravitational units and T_{ij} is the standard matter energy momentum tensor.

Contracting the above field equations, we have

$$F(R)R-\frac{n}{2}f(R)+(n-1)\,F(R)=8\pi T. \tag{2}$$

The Ricci scalar curvature function $f(R)$ can be expressed in terms of its derivative as under

$$f(R)=[-8\pi T+F(R)R+(n-1)\,\Box F(R)]/(n-3)\,. \tag{3}$$

Using this equation in (1), we obtain

$$[\,F(\,R\,)R\text{-}\Box F(\,R\,)\text{-}\,8\pi T\,]\,/\,n=[F(R)R_{ij}-\nabla_i\nabla_j\,F(R)-8\pi T_{ij}]/\,g_{ij}\,. \tag{4}$$

In the above equation, the expression on the left hand side is independent of the index i therefore the field equation (4) can be written as

$$A_i=[F(R)R_{ij}-\nabla_i\nabla_j\,F(R)-8\pi T_{ij}]/\,g_{ij}\,. \tag{5}$$

The paper is organized as under : In the section-2, we have presented non-vacuum static spherically symmetric field equations in *N*-dimension. The section-3 is devoted to discuss solution of the field equations by assuming constant scalar curvature. In the last section we summarize and discuss the results.

§ 2. Non-vacuum Static Spherically Symmetric field equations in *N*-dimension

In this section, we present non-vacuum static spherically symmetric field equations in V_n.

We consider the *N*-dimensional static spherically symmetric space-time

$$ds^2=A(r)dt^2-B(r)dr^2-r^2(d\theta_1^2+\sin^2\theta_1 d\theta_2^2+\sin^2\theta_1\sin^2\theta_2 d\theta_3^2+\cdots+\sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{n-3}d\theta_{n-2}^2). \tag{6}$$

Where A and B are the functions of radial coordinate r only.

Using equation (6) the non-vanishing components of second kind of Christoffel symbol are calculated as under

$$\Gamma^1_{11}=\frac{B'}{2B}\,,\qquad \Gamma^1_{22}=-\frac{r}{B}\,,\qquad \Gamma^1_{33}=\frac{-r\sin^2\theta_1}{B}\,,\qquad \Gamma^2_{33}=-\sin\theta_1\cos\theta_1\,,$$

$$\Gamma^1_{44}=\frac{-r\sin^2\theta_1\sin^2\theta_2}{B}\,,\qquad \Gamma^2_{44}=-\sin^2\theta_2\sin\theta_1\cos\theta_1\,,\qquad \Gamma^3_{44}=-\sin\theta_2\cos\theta_2$$

$$\Gamma^1_{55}=\frac{-r\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3}{B}\,,\qquad \Gamma^2_{55}=-\sin^2\theta_2\sin^2\theta_3\sin\theta_1\cos\theta_1\,,$$

$$\Gamma^3_{55}=-\sin^2\theta_3\sin\theta_2\cos\theta_2\,,\qquad \Gamma^4_{55}=-\sin\theta_3\cos\theta_3\,,$$

$$\Gamma^2_{12}=\Gamma^3_{13}=\Gamma^4_{14}=\Gamma^5_{15}=\cdots=\Gamma^{n-3}_{1(n-3)}=\Gamma^{n-2}_{1(n-2)}=\Gamma^{n-1}_{1(n-1)}=\frac{1}{r}\,,$$

$$\Gamma^3_{23}=\Gamma^4_{24}=\Gamma^5_{25}=\cdots=\Gamma^{n-3}_{2(n-3)}=\Gamma^{n-2}_{2(n-2)}=\Gamma^{n-1}_{2(n-1)}=\cot\theta_1\,,$$

$$\Gamma^4_{34}=\Gamma^5_{35}=\Gamma^6_{36}=\cdots=\Gamma^{n-2}_{3(n-1)}=\Gamma^{n-1}_{3(n-1)}=\cot\theta_2\,,$$

$$\Gamma^5_{45}=\Gamma^6_{46}=\cdots=\Gamma^{n-1}_{4(n-1)}=\cot\theta_3\,,$$

$$\Gamma^{1}_{nn} = \frac{A'}{2B}, \qquad \Gamma^{n}_{1n} = \frac{A'}{2A}. \tag{7}$$

The components of the Ricci tensor are calculated as

$$R_{11} = -\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{(n-2)B'}{2Br}, \tag{8}$$

$$R_{22} = -\frac{A'r}{2AB} + \frac{B'r}{2B^2} - \frac{(n-3)}{B} + (n-3), \tag{9}$$

$$R_{33} = Sin^2\theta_1 R_{22},$$

$$\vdots$$

$$R_{(n-3)(n-3)} = \sin^2\theta_{n-5} R_{(n-4)(n-4)}, \tag{10}$$

$$R_{(n-2)(n-2)} = \sin^2\theta_{n-4} R_{(n-3)(n-3)}, \tag{11}$$

$$R_{(n-1)(n-1)} = \sin^2\theta_{n-3} R_{(n-2)(n-2)}. \tag{12}$$

$$R_{nn} = \frac{A''}{2B} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{(n-2)A'}{2Br}. \tag{13}$$

The corresponding Ricci scalar is

$$R = \frac{A''}{AB} - \frac{A'^2}{2A^2B} - \frac{A'B'}{2AB^2} - \frac{(n-2)B'}{B^2r} + \frac{(n-2)A'}{ABr} + \frac{(n-2)(n-3)}{Br^2} - \frac{(n-2)(n-3)}{r^2} \tag{14}$$

where prime denotes derivative with respect to the radial coordinate r. The dust energy momentum tensor is defined as

$$T_{ij} = \rho u_i u_j \tag{15}$$

where $u_i = \delta_i^6$ is the *n*–velocity in co-moving coordinates and ρ is the density.

From the equation (5), $A_n - A_1 = 0$, $A_n - A_{(n-4)} = 0$, $A_n - A_{(n-3)(n-3)} = 0$, $A_n - A_{(n-2)} = 0$ and $A_n - A_{(n-1)} = 0$ respectively imply that

$$-\frac{F''}{B} + \frac{F'}{2B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + F\left[\frac{(n-2)}{2}\left(\frac{A'}{ABr} + \frac{2B'}{B^2r}\right)\right] - \frac{8\pi\rho}{A} = 0, \tag{16}$$

$$-\frac{F'}{Br} + \frac{F'A'}{2AB} + F\left(\frac{A''}{2AB} - \frac{A'^2}{4A^2B} - \frac{A'B'}{4AB^2} + \frac{(n-3)}{2}\frac{A'}{ABr} + \frac{B'}{2B^2r} - \frac{(n-3)}{Br^2} + \frac{(n-3)}{r^2}\right) - \frac{8\pi\rho}{A} = 0, \tag{17}$$

$$-\frac{F'}{Br} + \frac{F'A'}{2AB} + F\left(\frac{A''}{2AB} - \frac{A'^2}{4A^2B} - \frac{A'B'}{4AB^2} + \frac{(n-3)}{2}\frac{A'}{ABr} + \frac{B'}{2B^2r} - \frac{(n-3)}{Br^2} + \frac{(n-3)}{r^2}\right) - \frac{8\pi\rho}{A} = 0, \tag{18}$$

$$-\frac{F'}{Br} + \frac{F'A'}{2AB} + F\left(\frac{A''}{2AB} - \frac{A'^2}{4A^2B} - \frac{A'B'}{4AB^2} + \frac{(n-3)}{2}\frac{A'}{ABr} + \frac{B'}{2B^2r} - \frac{(n-3)}{Br^2} + \frac{(n-3)}{r^2}\right) - \frac{8\pi\rho}{A} = 0, \tag{19}$$

$$\vdots$$

$$-\frac{F'}{Br}+\frac{F'A'}{2AB}+F\left(\frac{A''}{2AB}-\frac{A'^2}{4A^2B}-\frac{A'B'}{4AB^2}+\frac{(n-3)}{2}\frac{A'}{ABr}+\frac{B'}{2B^2r}-\frac{(n-3)}{Br^2}+\frac{(n-3)}{r^2}\right)-\frac{8\pi\rho}{A}=0 . \quad (20)$$

It is interesting to note that, in *N*-dimensional case, we have only two independent non-linear differential equations with four unknown functions $F(r)$, $\rho(r)$, $A(r)$ and $B(r)$.

§ 3. Solution of the field equations by assuming constant scalar curvature

This section is devoted to study the non-trivial solution of the field equations in V_n by assuming constant scalar curvature. The conservation law of energy-momentum tensor, $T^{j}_{i;j}=0$, for dust matter implies that $A=\text{constant}=A_0\,(\text{say})$. Thus the system of field equations (16), (17), (18), (19) and (20) is reduced to three unknowns $F(r), \rho(r)$ and $B(r)$ with the following two independent non-linear differential equations

$$-\frac{F''}{B}+\frac{F'B'}{2B^2}+F\left(\frac{(n-2)}{2}\frac{B'}{B^2r}\right)-\frac{8\pi\rho}{A_0}=0, \quad (21)$$

$$-\frac{F'}{Br}+F\left(\frac{B'}{2B^2r}-\frac{(n-3)}{Br^2}+\frac{(n-3)}{r^2}\right)-\frac{8\pi\rho}{A_0}=0. \quad (22)$$

Thus we get only two independent non-linear differential equations in three unknown.

Hence we discuss the solution of the field equations by assuming constant scalar curvature $(R=R_0)$, i.e., $F(R_0)=\text{constant}$. Therefore above field equations become

$$F(R_0)\left(\frac{(n-2)}{2}\frac{B'}{B^2r}\right)-\frac{8\pi\rho}{A_0}=0, \quad (23)$$

$$F(R_0)\left(\frac{B'}{2B^2r}-\frac{(n-3)}{Br^2}+\frac{(n-3)}{r^2}\right)-\frac{8\pi\rho}{A_0}=0. \quad (24)$$

In this way we have two differential equations with two unknown, $B(r)$ and $\rho(r)$. From equation (23) and (24), we have an ordinary differential equation in terms of $B(r)$ such that

$$B'r+2B-2B^2=0. \quad (25)$$

After solving the equation (25), we have a solution

$$B(R)=\frac{1}{1-c_1r^2} \quad (26)$$

where c_1 is a constant.
Putting this value of *B* in any of the above equations, we have

$$\rho=\frac{(n-2)c_1A_0F(R_0)}{8\pi}=\rho_0 \quad (27)$$

which is nothing but a constant.
Hence the space-time for constant curvature solution takes the following form

$$ds^2 = A_0(r)dt^2 - \frac{1}{1-c_1r^2}dr^2 - r^2(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{n-3}d\theta_{n-2}^2) \tag{28}$$

which is required solution.

This solution corresponds to the well-known Tolman-Oppenheimer-Volkoff (TOV) space-time when density is constant and pressure is neglected.

The scalar curvature becomes $R_0 = -(n-1)(n-2)c_1$ and therefore, equation (3), yields

$$f(R_0) = \frac{-8\pi\rho_0}{A_0} + F(R_0)R_0 . \tag{29}$$

Putting the value of ρ_0 and R_0, it follows that

$$f(R_0) = -2(n-2)c_1 f'(R_0) . \tag{30}$$

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§ 4. *Concluding Remark* :

In this paper, we have investigated *N*-dimensional dust static spherically symmetric non-vacuum solutions in $f(R)$ theory of gravity with the assumption of constant scalar curvature. The scalar curvature for this solution obtained as non-zero constant. Therefore, this leads to constant density of dust matter and corresponds to well known Tolman-Oppenheimer-Volkoff space-time when density is constant and pressure is neglected.

It is interesting to investigate solutions for non-static space-times with energy-momentum tensor of other types of fluid.

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