LEGENDRE B-SPLINE COLLOCATION: A NOVEL TECHNIQUE TO SOLVE CONVECTION DIFFUSION PROBLEM

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Abstract:
In this study, convection diffusion problems—a partial integral differential equation that arises in a variety of processes including heat conduction in materials with memory, population dynamics, viscoelasticity, etc.—are solved using the Legendre B Spline basis. We used time discretization with the Legendre Cubic B-spline collocation method to solve Convection Diffusion problem. The two-point Euler backward difference approach is used to discretize the problem and the Legendre B-Spline Collocation method is applied to identify the unknown function. Gauss Elimination method is implemented in MATLAB to solve the system of algebraic equations obtained. The procedure is demonstrated by a numerical solution of Convection Diffusion equation.
Keywords: Convection Diffusion; Partial Integro-differential equation; Legendre B-spline; collocation; Discretization.

1 Introduction

There are several instances in diverse fields of physics, biology, and engineering that, when formally described, result in partial Integro differential equation systems with time and space variables. The integral term containing the unknown function acts as a system memory effect in partial Integro differential equations. Due to this partial Integro differential equation, certain significant and relevant physical circumstances in several fields, such as convection-diffusion issues, electromagnetic theory, nuclear reactor dynamics, plasma physics, and geophysics, can be described.

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection.

Convection is the term for the molecular motion-induced heat transfer across a fluid. Convection is the circular motion that develops as warmer, less dense air or liquid rises and the cooler, slower-moving air or liquid descends due to its faster-moving molecules.

Convection may be used in a variety of scientific or engineering contexts or applications, each with slightly different but related definitions. In a broader sense, convection refers to fluid motion regardless of the cause in the field of fluid mechanics. But in thermodynamics, the term "convection" frequently refers only to convection-based heat transfer. In both living and non-living systems, diffusion is a frequent and significant activity. Transport that is done passively is called diffusion.

Diffusion is the term used to describe the net movement of a substance caused by random molecular mobility from a location of higher concentration to a region of lower concentration. For instance, oxygen molecules can pass through cell membranes and enter a cell if there is a larger quantity of oxygen outside the cell than inside.

For applications like water quality modelling, air pollution, meteorology, oceanography, and other physical disciplines, it is important to compute the transit of fluid characteristics or trace ingredient concentrations within a fluid. Partial Integro-differential equations are present in many mathematical descriptions of physical events like convection (advection)-diffusion. This section attempts to obtain an approximation of the solution to the convection-diffusion Integro-differential equation with a weakly singular kernel. This convective diffusion equation is frequently utilized in many different fields.

We have attempted to solve partial Integro differential equation of the type

$$u_t(x,t) + mu_x(x,t) - bu_{xx}(x,t) = \int_0^t (t-s)^{-\alpha} u(x,s)ds + f(x,t) \quad (1)$$

$x \in [0,1] \quad \alpha = \frac{1}{2} \quad t > 0 \quad m = 0.05 \quad b = 0.4$

With initial condition $u(x,0) = sin\pi x \quad 0 \leq x \leq 1$

And boundary condition $u(0,t) = 0 = u(1,t) \quad 0 \leq t \leq T$

which is a representation of convection diffusion process in many phenomena.

Partial Integro differential equations haven't been solved very often in the past. Several notable pieces include Central difference and cubic B-Spline by Shahid S. Siddiqi and Saima Arshed [1] sixth degree B-Spline by A. F. Soliman et. all, [2], Laplace transform by Jyoti Thorwe [3], septic B-Spline collocation, and backward differences by A. Ali fazal-i-haq [4], Cubic B-Spline collocation method Mandeep Kaur [5]

In addition, this article is completed in the manner described below. In section 2, Euler Backward discretization scheme with the Legendre B-spline collocation method is described. In section 3, The implementation of the suggested method is discussed through the numerical examples.

2 Time discretization and Legendre B-Spline Collocation

Here the time derivative is discretized by the first order backward Euler scheme. Let $t_n = nk$ where k is the time step, $u^n(x)$ approximates the value of u (x, t) at a time point $t = t_n, n = 0,1, \dots$

Considering the temporal discrete process of equation (1) at time point $t = t_{n+1}$, the first expression in left side of equation (1) is approximated by

$$u_t(x,t_{n+1}) \approx \frac{u(x,t_{n+1}) - u(x,t_n)}{k} \quad (2)$$

Substitute (2) in (1) we get

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{k} + mu_x(x,t_{n+1}) - bu_{xx}(x,t_{n+1}) = f(x,t) + \int_0^{t_{n+1}} (t-s)^{-\frac{1}{2}} u(x,s)ds \quad (3)$$

Integration on R.H.S of (3) is further simplify using change of variable, we get the following form of equation (3)

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{k} + mu_x(x,t_{n+1}) - bu_{xx}(x,t_{n+1}) = f(x,t) + \frac{k^{1-\alpha}}{1-\alpha}\sum_{j=0}^{n} u(x,t_{n-j+1})\left[(j+1)^{\frac{1}{2}} - j^{\frac{1}{2}}\right]$$

Let us denote $u(x,t_{n+1}) = u^{n+1}(x)$ and $b_j = (j+1)^{\frac{1}{2}} - j^{\frac{1}{2}} \quad j = 1,2,3, \dots$

So, above equation is written as

$$u^{n+1}(x) + mu_x^{n+1}(x) - kbu_{xx}^{n+1}(x) - 2k^{\frac{3}{2}}u^{n+1}(x) = kf(x,t) + u^n(x) + 2k^{\frac{3}{2}}\sum_{j=1}^{n} b_j\, u^{n-j+1}(x), \quad n \geq 1$$

(4)

For the special case n = 0 which is the first-time step, the scheme simply leads to

$$u^1(x) + mu_x^1(x) - bku_{xx}^1(x) - 2k^{\frac{3}{2}}u^1(x) = f(x,t) + u^0(x) \qquad (5)$$

Where right hand side we can get from the given initial conditions and left-hand side unknown u(x) can be determine using Legendre B-spline collocation method. The Equation (5) will be collocated at the intermediate points and to determine all the coefficient we will use boundary conditions.

Taking u(x) as

$$u(x) = \sum_{i=1}^{n+1} c_i B_{i,k}(x) \qquad 2 \leq k \leq n+1$$

(6)

where c_i are the coefficients to be determined and $B_{i,k}(t)$ are the Legendre B-spline basis functions [8]. The order 'k' of the basis function results in a polynomial of degree n, where n = k-1. As we are considering here Legendre Cubic B-spline, take n = 3

So

$$B_{04} = \frac{1}{6}\left(-\frac{5}{2}x^3 + \frac{9}{2}x^2 - \frac{3}{2}x - \frac{1}{2}\right)$$

$$B_{14} = \frac{1}{6}\left(\frac{15}{2}x^3 - 9x^2 - \frac{3}{2}x + 7\right)$$

$$B_{24} = \frac{1}{6}\left(-\frac{15}{2}x^3 + \frac{3}{2}x^2 + \frac{9}{2}x - \frac{1}{2}\right)$$

$$B_{34} = \frac{1}{6}\left(\frac{1}{2}(5x^3 - 3x)\right)$$

3. Numerical Example:

Example 1:

Consider the partial Integro differential equation [Shahid S. Siddiqi (2014) [1]]

$$u_t(x,t) + mu_x(x,t) - bu_{xx}(x,t) = \int_0^t (t-s)^{-\alpha} u(x,s)ds + f(x,t)$$

(1)

$x \in [0,1] \quad \alpha = \frac{1}{2} \quad t > 0 \quad m = 0.05 \quad b = 0.4$

With initial condition $u(x,0) = \sin\pi x \quad 0 \leq x \leq 1$

And boundary condition $u(0,t) = 0 = u(1,t) \quad 0 \leq t \leq T$

Solution:

Using boundary conditions and equation (6), we obtain

$$u(0,t) = 0 \Rightarrow -\frac{c_1}{12} + \frac{7c_2}{6} - \frac{c_3}{12} = 0 \qquad (7)$$

And

$$u(1,t) = 0 \Rightarrow \frac{c_2}{6} + \frac{c_3}{6} + \frac{c_4}{6} = 0 \qquad (8)$$

Solving equation (5) for intermediate values using (6) and considering equation (7) and (8) we will get system of four linear equations in four unknowns, which can be solve by Gauss elimination method gives first initial state vector $C^0 = [c_1^0,\ c_2^0,\ c_3^0,\ c_4^0]^T$

Solving the system,

$c_1^0 = -5.8573, \quad c_2^0 = -0.3254, \quad c_3^0 = 1.3016, \quad c_4^0 = -1.9762$

Substituting these values in equation (6) we can get required unknown function $u^1(x)$

$$u^1(x) = -5.8573B_{04} - 0.3254B_{14} + 1.3016B_{24} - 1.9762B_{34} \quad (9)$$

Following is the comparison with exact solution for t = 0 is $u(x,t) = \sin(\pi x)$

x:	**Numerical Solution:**	**x:**	**Numerical Solution:**
0	0	0.6	0.9191
0.1	0.3467	0.7	0.7838
0.2	0.6194	0.8	0.5594
0.3	0.8154	0.9	0.2434
0.4	0.9324	1	0
0.5	0.9678		

For the next time step substitute n = 1 in equation (4) to get

$$u^2(x) + mu_x^2(x) - bku_{xx}^2(x) - 2k^{\frac{3}{2}}u^2(x) = f(x,t) + u^1(x) + 2k^{\frac{3}{2}}b_1u^1(x) \quad (10)$$

where $b_1 = 2^{\frac{1}{2}} - 1$ and $u^1(x)$ is according to equation (9)

Solving equation (10) for intermediate values using (6) and considering equation (7) and (8), we will get the system of four linear equations in four unknowns, solving it we obtain the second state vector $C^1 = [c_1^1,\ c_2^1,\ c_3^1,\ c_4^1]^T$

we are getting

$c_1^1 = -6.4273,\ c_2^1 = -0.3737,\ c_3^1 = 1.1954,\ c_4^1 = -0.8216$

Substituting these values in equation (6) we can get required unknown function $u^2(x)$.

$$u^2(x) = -6.4273B_{04} - 0.3737B_{14} + 1.1954B_{24} - 0.8216B_{34} \quad (11)$$

Following is the comparison with exact solution for t = 0.01:

x:	**Numerical Solution:**	**x:**	**Numerical Solution:**
0	0	0.6	0.8069
0.1	0.3194	0.7	0.6982
0.2	0.5619	0.8	0.5260
0.3	0.7296	0.9	0.2925
0.4	0.8249	1	0
0.5	0.8499		

For the next time step substitute n = 2 in equation (4) we get

$$u^3(x) + mu_x^3(x) - bku_{xx}^3(x) - 2k^{\frac{3}{2}}u^3(x) = f(x,t) + u^2(x) + 2k^{\frac{3}{2}}b_1u^2(x) + 2k^{\frac{3}{2}}b_2u^1(x) \quad (12)$$

where $b_1 = 2^{\frac{1}{2}} - 1,\ b_2 = 3^{\frac{1}{2}} - 2^{\frac{1}{2}}$ and $u^2(x)$ is according to equation (11) and $u^1(x)$ according to equation (9)

Solving equation (12) for intermediate values using (6) and considering equation (7) and (8) we obtain the third state vector $C^2 = [c_1^2,\ c_2^2,\ c_3^2,\ c_4^2]^T$

we are getting

$c_1^2 = -4.9877,\ c_2^2 = -0.2777,\ c_3^2 = 1.0997,\ c_4^2 = -0.8220$

Substituting these values in equation (6) we get required unknown function $u^3(x)$.

$$u^3(x) = -4.9877B_{04} - 0.2777B_{14} + 1.0997B_{24} - 0.8220B_{34} \quad (13)$$

Following is the comparison with exact solution for t = 0.02:

x:	Numerical Solution:	x:	Numerical Solution:
0	0	0.6	0.7265
0.1	0.2731	0.7	0.6354
0.2	0.4852	0.8	0.4839
0.3	0.6366	0.9	0.2721
0.4	0.7272	1	0
0.5	0.7571		

Example 2:

Consider the partial Integro differential equation [Shahid S. Siddiqi (2014) [1]]

$$u_t(x,t) + mu_x(x,t) - bu_{xx}(x,t) = \int_0^t (t-s)^{-\alpha} u(x,s)ds + f(x,t)$$

$x \in [0,1] \quad \alpha = \frac{1}{4} \quad t > 0 \quad m = 0.5 \quad b = 0.001$

With initial condition $u(x,0) = 2sin^2 \pi x \quad 0 \le x \le 1$

And boundary condition $u(0,t) = 0 = u(1,t) \quad 0 \le t \le T$

Solution:

Following the same process as in previous section we get the solutions as follows for initial state vector

$c_1^0 = -10.0826, \quad c_2^0 = -0.5601, \quad c_3^0 = 2.2406, \quad c_4^0 = -1.6804$

x:	Numerical Solution:	x:	Numerical Solution:
0	0	0.6	1.4788
0.1	0.5546	0.7	1.2940
0.2	0.9859	0.8	0.9859
0.3	1.2940	0.9	0.5546
0.4	1.4788	1	0
0.5	1.5404		

For i = 1

$c_1^0 = -8.8098, \quad c_2^0 = -0.4805, \quad c_3^0 = 2.0828, \quad c_4^0 = -1.6023$

x:	Numerical Solution:	x:	Numerical Solution:
0	0	0.6	1.3650
0.1	0.5028	0.7	1.1986
0.2	0.8971	0.8	0.9164
0.3	1.1817	0.9	0.5173
0.4	1.3554	1	0
0.5	1.4169		

For i = 2

$c_1^0 = -8.6119, \quad c_2^0 = -0.4799, \quad c_3^0 = 1.8929, \quad c_4^0 = -1.4129$

x:	**Numerical Solution:**	**x:**	**Numerical Solution:**
0	0	0.6	1.2509
0.1	0.4706	0.7	1.0939
0.2	0.8361	0.8	0.8329
0.3	1.0967	0.9	0.4682
0.4	1.2525	1	0
0.5	1.3039		

3. Conclusion:

The Legendre B-Spline collocation technique is essentially new. It has been used on a challenging problem like viscoelastic, for which obtaining a numerical solution is essential. It was an effort to discover a different approach to solve viscoelastic problem.

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