# GLOBAL APPROXIMATION BY A NEW FAMILY OF BETA OPERATORS

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**Abstract**
**In 1985, Upreti studied approximation properties of beta operators. Later in 1991, Zhou obtained direct and inverse theorems for these operators. In the last decade, Gupta and Dogru obtained some direct results for beta operators. In the present paper, we establish a global direct theorem for a new family of beta operators, by using Ditzian-Totik modulus of smoothness.**

**Key Words: Global approximation, Linear positive operators, Beta operators, K-functional, Ditzian-Totik modulus of smoothness.**

***MSC* : 41A17, 41A36.**

## I. INTRODUCTION

Felten [3] obtained local and global approximation theorems for positive linear operators. In the last decade, Finta [4] studied direct local and global approximation theorems for Baskakov type operators and Szasz-Mirakian type operators. Recently, Kumar [12] studied direct results for Beta-Szasz operators in simultaneous approximation. Beta operators have been studied by several researchers [5, 6, 7, 8, 9, 10, 13, 14]. In the present paper, we establish a global direct theorem for a new family of beta operators introduced by Gupta et al. [9] to approximate lebesgue integrable functions on $[0,\infty)$ as

$$B_n(f,x)=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}b_{n,v}(t)\,f(t)\,dt, \qquad (1.1)$$

Where $n\in N$ (the set of natural numbers),

$x\in[0,\infty)$, $b_{n,v}(t)=\dfrac{1}{\beta(v,n+1)}t^{v-1}(1+t)^{-n-v-1}$

and

$\beta(v,n+1)=(v-1)!n!/(n+v)!$ the beta function.

Some **basic properties of** $b_{n,v}(x)$ are as follows

(i). $\int_0^{\infty}b_{n-\mathrm{r},v+\mathrm{r}}(t)\;dt=1$ (1.2)

(ii). $\int_0^{\infty}t\,b_{n,v}(t)\;dt=\dfrac{v}{n}$ (1.3)

(iii). $\int_0^{\infty}t^2\,b_{n,v}(t)\;dt=\dfrac{v(v+1)}{n(n-1)}$ (1.4)

(iv). $\sum_{v=1}^{\infty}b_{n,v}(x)=(n+1)$ (1.5)

(v). $\sum_{v=1}^{\infty}v\,b_{n,v}(x)=(n+1)[1+(n+2)x]$ (1.6)

(vi). $\sum_{v=1}^{\infty}v^2\,b_{n,v}(x)=(n+1)[1+3(n+2)x$
$+(n+2)(n+3)x^2]$ (1.7)

(vii). $x(1+x)b'_{n,v}(x)=[(v-1)-(n+2)x]$
$\times b_{n,v}(x)$ (1.8)

where $n\in N$ and $x\in[0,\infty)$.

It can be easily verified that the operators (1.1) are linear positive operators and the order of approximation by these operators is at best $O(n^{-1})$ as $n\to\infty$, howsoever smooth the function may be.

Let $£_1^{\mathrm{r}}[0,\infty)$ be the class of functions $g$ defined by

$£_1^{\mathrm{r}}[0,\infty)=\{g:g^{(\mathrm{r})}\in L_1[0,a]$,

for every $a\in(0,\infty)$ and $\left|g^{(\mathrm{r})}(t)\right|\le M(1+t)^m\}$

Where the constants $M$ and $m$ depend on $g$, and $L_p[a,b], 0<p<\infty$ stands for the $L_p$-space.

It is obvious that $L_p^r[0,\infty)$ is not contained in $£_1^r[0,\infty)$.

Due to Ditzian and Totik [2], the modulus of smoothness of a function *f* is defined by

$$\omega_\phi^2(f,t)=\sup_{0<h\le t}\left\| \Delta_{h\phi}^2 f \right\|_p \tag{1.9}$$

where $\phi^2(x)=x(1+x)$ and

$$\Delta_h^2 f(x)= f(x-h)-2f(x)+f(x+h) \quad \text{if } [x-h,x+h]\subset[0,\infty)$$

otherwise $\Delta_h^2 f(x)=0$.

Let

$$\overline{W}_p^2\left(\phi,[0,\infty)\right)=\left\{ g\in L_p[0,\infty): g'\in AC_{loc}[0,\infty) \text{ and } \phi^2 g''\in L_p[0,\infty)\right\}.$$

Following [2], it can be easily verified that the modulus of smoothness defined by (1.9) , is equivalent to the modified *K*-functional given by

$$\overline{K}_\phi^2\left(f,t^2\right)=\inf\left\{ \left\| f-g\right\|_p + t^2\left\|\phi^2 g''\right\|_p + t^4\left\| g''\right\|_p : g\in \overline{W}_p^2\left(\phi,[0,\infty)\right)\right\}.$$

The main object of the present paper is to establish a global approximation result for the operators (1.1) in terms of Ditzian-Totik modulus of second order.

## II. PRELIMINARY RESULTS

This section consists of some auxiliary results, which will be helpful in proving the main results of next section.

**Lemma 2.1.** For $m,r\in N^0$ (the set of non-negative integers), let the function $T_{r,n,m}(x)$ be defined as

$$T_{r,n,m}(x)=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty} b_{n+r,v}(x) \times\int_0^\infty b_{n-r,v+r}(t)\,(t-x)^m\,dt .$$

Then

$$T_{r,n,0}(x)=1,\ T_{r,n,1}(x)=\frac{(1+r)(1+2x)}{(n-r)} \quad (n>r),$$

and for all $n>m+r$, there holds the recurrence relation

$$(n-m-r)\,T_{r,n,m+1}(x)=x(1+x)[T'_{r,n,m}(x) + 2mT_{r,n,m-1}(x)]+(m+r+1)(1+2x)\,T_{r,n,m}(x)$$

Consequently, for each $x\in[0,\infty)$, we have

$$T_{r,n,m}(x)=O\left(n^{-[(m+1)/2]}\right),$$

where $[\alpha]$ denotes the integral part of $\alpha$.

**Proof.** Using the definition of $T_{r,n,m}(x)$ and basic properties of $b_{n,v}(x)$, we obtain

$$T_{r,n,0}(x)=1 \quad \text{and}$$

$$T_{r,n,1}(x)=\frac{(1+r)(1+2x)}{(n-r)}$$

Now, we have

$$x(1+x)T'_{r,n,m}(x)=\frac{x(1+x)}{(n+r+1)}\sum_{v=1}^{\infty} b'_{n+r,v}(x) \times\int_0^\infty b_{n-r,v+r}(t)\,(t-x)^m\,dt - mx(1+x)T_{r,n,m-1}(x)$$

Therefore, using (1.8) we get

$$x(1+x)[T'_{r,n,m}(x)+mT_{r,n,m-1}(x)]$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty}[(v-1)-(n+r+2)x](x) \times\ b_{n+r,v}\int_0^\infty b_{n-r,v+r}(t)\,(t-x)^m\,dt$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty} b_{n+r,v}(x) \times\int_0^\infty \left[\left\{(v+r-1)-(n-r+2)\,t\right\}+(n-r+2)(t-x) \times\ +r(1+2x)\right] b_{n-r,v+r}(t)\,(t-x)^m\,dt$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty} b_{n+r,v}(x) \times\int_0^\infty t(1+t)\,b'_{n-r,v+r}(t)\,(t-x)^m\,dt +(n-r+2)\,T_{r,n,m+1}(x)-r(1+2x)\,T_{r,n,m}(x)$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x)$$
$$\times\int_0^{\infty}[(t-x)^2+(1+2x)(t-x)+x(1+x)]$$
$$\times b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt$$

$$+(n-r+2)T_{\mathrm{r},n,m+1}(x)-r(1+2x)T_{\mathrm{r},n,m}(x)$$

$$=\frac{1}{(n+r+1)}\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x)$$

$$\times\Bigg[\int_0^{\infty} b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^{m+2}\,dt+(1+2x)$$
$$\times\int_0^{\infty} b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^{m+1}dt$$
$$+x(1+x)\int_0^{\infty} b'_{n-\mathrm{r},v+\mathrm{r}}(t)\,(t-x)^m\,dt\Bigg]$$

$$+(n-r+2)T_{\mathrm{r},n,m+1}(x)-r(1+2x)T_{\mathrm{r},n,m}(x)$$

$$=-(m+2)T_{\mathrm{r},n,m+1}(x)-(m+1)(1+2x)T_{\mathrm{r},n,m}(x)-mx(1+x)T_{\mathrm{r},n,m-1}(x)$$

$$+(n-r+2)T_{\mathrm{r},n,m+1}(x)-r(1+2x)T_{\mathrm{r},n,m}(x)$$

Thus, collecting the like terms, we get the desired recurrence relation

$$(n-m-r)T_{\mathrm{r},n,m+1}(x)$$

$$=x(1+x)[T'_{\mathrm{r},n,m}(x)+2mT_{\mathrm{r},n,m-1}(x)]+(m+r+1)(1+2x)T_{\mathrm{r},n,m}(x)$$

The other consequence easily follows from the above recurrence relation.

**Corollary 2.2.** It can be easily verified from Lemma 2.1 that

$$T_{\mathrm{r},n,2m}(x)=\sum_{j=0}^{m} p_{j,m,n}(x)\left[\frac{x(1+x)}{n}\right]^{m-j} n^{-2j}$$
and
$$T_{\mathrm{r},n,2m+1}(x)=(1+2x)\sum_{j=0}^{m} q_{j,m,n}(x)\left[\frac{x(1+x)}{n}\right]^{m-j} n^{-(2j+1)}$$

where $p_{j,m,n}(x)$ and $q_{j,m,n}(x)$ are certain polynomials in $x$ of fixed degree with coefficients which are bounded uniformly for all $n$.

**Corollary 2.3**. For every $m\in N^0$, $n>r+2m$ and $x\in[0,\infty)$ there exist constants $C_1$ and $C_2$ such that

$$\left|B_{n,\mathrm{r}}\left((t-x)^{2m},\,x\right)\right|\le C_1 n^{-m}\left[x(1+x)+n^{-1}\right]^m \quad (2.1)$$

and

$$\left|B_{n,\mathrm{r}}\left((t-x)^{2m+1},\,x\right)\right|\le C_2(1+2x)\,n^{-(m+1)}$$
$$\times\left[x(1+x)+n^{-1}\right]^m \quad (2.2)$$

Consequently, for fixed $x\in[0,\infty)$ and $n\to\infty$, we have

$$\left|B_{n,\mathrm{r}}\left((t-x)^{2m},\,x\right)\right|=O\left(n^{-[(m+1)/2]}\right) \quad (2.3)$$

**Proof.** Noticing the fact that $B_{n,\mathrm{r}}\left((t-x)^m,\,x\right)=\frac{\alpha(n,r)(n+r+1)}{(n+1)}T_{\mathrm{r},n,m}(x)$, the estimates (2.1) and (2.2) follow from Corollary 2.2 along the lines of [11]. The other consequence (2.3) easily follows from (2.1) and (2.2).

**Lemma 2.4.** For $x,t\in[0,\infty)$ and $n>m+r$, there exists a constant $C$ independent of $n$ such that

$$B_{n,\mathrm{r}}\left((1+t)^{-m},\,x\right)=C\,(1+x)^{-m}.$$

**Proof.** By straight computations we have

$$(1+t)^{-m}\,b_{n-\mathrm{r},v+\mathrm{r}}(t)=\prod_{j=1}^{\mathrm{m}}\frac{(n-r+j)}{(n+v+j)}b_{n-\mathrm{r+m},v+\mathrm{r}}(t)$$

and

$$b_{n+\mathrm{r},v}(x)=(1+x)^{-m}\prod_{j=0}^{\mathrm{m-1}}\frac{(n+r+v-j)}{(n+r-j)}b_{n+\mathrm{r-m},v}(x)$$

.

Making use of the above two identities and basic properties of $b_{n,v}(x)$, we have

$$B_{n,\mathrm{r}}\left((1+t)^{-m},\,x\right)=\frac{\alpha(n,r)}{(n+1)}\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x)$$
$$\times\int_0^{\infty} b_{n-\mathrm{r},v+\mathrm{r}}(t)\,(1+t)^{-m}\,dt$$
$$=\frac{\alpha(n,r)}{(n+1)}\sum_{v=1}^{\infty} b_{n+\mathrm{r},v}(x)\prod_{j=1}^{\mathrm{m}}\frac{(n-r+j)}{(n+v+j)}$$
$$\times\int_0^{\infty} b_{n-\mathrm{r+m},v+\mathrm{r}}(t)\,dt$$

$$=\frac{\alpha(n,r)}{(n+1)}(1+x)^{-m}\sum_{v=1}^{\infty}b_{n+\mathrm{r}-\mathrm{m},v}(x)$$
$$\times\prod_{j=0}^{\mathrm{m}-1}\frac{(n+r+v-j)}{(n+r-j)}\prod_{j=1}^{\mathrm{m}}\frac{(n-r+j)}{(n+v+j)}\leq C(1+x)^{-m}$$

**Corollary 2.5.** Let $e_0$ and $e_1$ be two monomials. Then for $x\in[0,\infty)$ and $n\to\infty$, we have

$B_{n,\mathrm{r}}\left(e_0,\ x\right)=1+O(n^{-1})$ and

$B_{n,\mathrm{r}}\left(e_1,\ x\right)=x[1+O(n^{-1})]$.

**Lemma 2.6.** Let the function $F_n(y)$ be defined by

$$F_n(y)=\left[\int_0^{\infty}\int_0^{y}-\int_0^{y}\int_0^{\infty}\right]\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)b_{n-\mathrm{r},v+\mathrm{r}}(t)\times(y-t)\,dt\,dx$$

Then there exists a constant *C* independent of *n* and *y* such that

$$F_n(y)\leq C\,y(1+y)/n.$$

**Proof.** Using basic properties of $b_{n,v}(x)$ and proceeding in the same way as in Lemma 5.2[1], the above lemma easily follows.

## III. GLOBAL APPROXIMATION

In this section, we establish the following global direct result.

**Theorem 3.1.** For the function $f\in L_p[0,\infty)$, $1\leq p<\infty$ and $n>r+4$, there exists a constant *C* independent of *n* such that

$$\left\|B_{n,\mathrm{r}}f-f\right\|_p\leq C\left[\omega_{\phi}^2(f,n^{-1/2})+n^{-1}\|f\|_p\right].$$

**Proof.** By Taylor's expansion of $\mathrm{g}$, we have

$$\mathrm{g}(t)=\mathrm{g}(x)+(t-x)\mathrm{g}'(x)+\int_x^t(t-y)\mathrm{g}''(y)\,dy \qquad (3.1)$$

Since $B_{n,\mathrm{r}}(f,x)$ are uniformly bounded operators, therefore, for every $\mathrm{g}\in\overline{W}_p^2\left(\phi,[0,\infty)\right)$, we have

$$\left\|B_{n,\mathrm{r}}f-f\right\|_p\leq C_1\left\|f-\mathrm{g}\right\|_p+\left\|B_{n,\mathrm{r}}\mathrm{g}-\mathrm{g}\right\|_p \quad (3.2)$$

Using Corollaries 2.3, 2.5 and the above expansion of *g(t)* together with the following [2], we obtain

$$\left\|B_{n,\mathrm{r}}\mathrm{g}-\mathrm{g}\right\|_p\leq C_2\left(\left\|\mathrm{g}\right\|_p+\left\|\mathrm{g}'\right\|_{L_p[0,1]}\right)$$
$$+\left\|(1+2x)\mathrm{g}'\right\|_{L_p[1,\infty)}+\left\|B_{n,\mathrm{r}}\left(R(\mathrm{g},t,x),\ x\right)\right\|_p$$
$$\leq C_3n^{-1}\left(\left\|\mathrm{g}\right\|_p+\left\|\phi^2\mathrm{g}''\right\|_p\right)+\left\|B_{n,\mathrm{r}}\left(R(\mathrm{g},t,x),\ x\right)\right\|_p \qquad (3.3)$$

where $R(\mathrm{g},t,x)=\int_x^t(t-y)\,\mathrm{g}''(y)\,dy$.

Now for $1\leq p\leq\infty$, we establish the following inequality

$$\left\|B_{n,\mathrm{r}}\left(R(\mathrm{g},t,x),\ x\right)\right\|_p\leq C_4n^{-1}\left\|(\phi^2+n^{-1})\mathrm{g}''\right\|_p \qquad (3.4)$$

By Riesz-Thorin theorem, to derive (3.4), we only need to prove two cases of $p=1$ and $p=\infty$.

Following [11], the case $p=\infty$ easily follows by using Corollary 2.3 for $m=1$ and Lemma 2.4.

To derive (3.4) for the case $p=1$, we use Fubini's theorem twice, the definition of $F_n(y)$ and Lemma 2.6 as

$$\int_0^{\infty}\left|B_{n,\mathrm{r}}\left(R(\mathrm{g},t,x),\ x\right)\right|dx\leq\frac{\alpha(n,r)}{(n+1)}\int_0^{\infty}\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)$$
$$\times\int_0^{\infty}b_{n-\mathrm{r},v+\mathrm{r}}(t)\left|\int_x^t(t-y)\mathrm{g}''(y)\,dy\right|dt\,dx$$
$$=\frac{\alpha(n,r)}{(n+1)}\int_0^{\infty}\left|\mathrm{g}''(y)\right|\left[\int_0^{\infty}\int_0^{y}-\int_0^{y}\int_0^{\infty}\right](y-t)$$
$$\times\sum_{v=1}^{\infty}b_{n+\mathrm{r},v}(x)b_{n-\mathrm{r},v+\mathrm{r}}(t)\,dt\,dx\,dy$$
$$=\frac{\alpha(n,r)}{(n+1)}\int_0^{\infty}\left|\mathrm{g}''(y)\right|F_n(y)\,dy\leq C_5n^{-1}\left\|\phi^2\mathrm{g}''\right\|_1$$
$$\leq C_6n^{-1}\left\|(\phi^2+n^{-1})\mathrm{g}''\right\|_1$$

where the constant $C_6$ is independent of *n*.

Thus by Riesz-Thorin theorem, inequality (3.4) holds for all $1\leq p\leq\infty$.

Next collecting the estimates of (3.2), (3.3) and (3.4), we get

$$\left\| B_{n,r} f - f \right\|_p \leq C_1 \left\| f - g \right\|_p$$
$$+ C_3 n^{-1} \left( \left\| f - g \right\|_p + \left\| f \right\|_p + \left\| \phi^2 g'' \right\|_p \right)$$
$$+ C_4 n^{-1} \left\| (\phi^2 + n^{-1}) g'' \right\|_p$$
$$\leq C_7 \left( \left\| f - g \right\|_p + n^{-1} \left\| \phi^2 g'' \right\|_p \right.$$
$$\left. + n^{-2} \left\| g'' \right\|_p + n^{-1} \left\| f \right\|_p \right) \qquad (3.5)$$

Finally, taking the infimum over all $g \in \overline{W}_p^2(\phi, [0,\infty))$ on the right hand side of (3.5), we obtain

$$\left\| B_{n,r} f - f \right\|_p \leq C \left( \overline{K}_\phi^2 (f, n^{-1}) + n^{-1} \left\| f \right\|_p \right).$$

This completes the proof of Theorem 3.1.

**Remark 3.2.** It is important to note that the conclusion of the above Theorem 3.1 is also true on the spaces $L_p[0,\infty)$, $1 \leq p < \infty$ i.e.

$\lim_{n\to\infty} \left\| B_{n,r} f - f \right\|_p = 0$ for all $f \in L_p[0,\infty)$,

because the most basic fact about $\omega_\phi^2(f, n^{-1})$ is that

$\lim_{n\to\infty} \omega_\phi^2(f, n^{-1}) = 0$ for all $f \in L_p[0,\infty)$ and $1 \leq p < \infty$,

or for all bounded functions $f \in C[0,\infty)$ satisfying

$\lim_{x\to\infty} f(x) = L_\infty < \infty$ if $p = \infty$ (cf. [2, p.36]).

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