DIVISION OF MATRICES AND SYSTEM OF DIOPHANTINE EQUATIONS

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Abstract
This paper deals with the division of non-zero, non-square matrices.

Introduction

Every under-graduate student of Mathematics is familiar with the creation of a mathematical model known as matrices and its various operations. Normally, in the textbooks of Algebra, the division of two non-zero, non-singular matrices is considered. In other words, if A and B are two non-zero, non-singular square matrices of order n then the division $\frac{A}{B}$ is considered as follows: $\frac{A}{B} = A.\frac{1}{B} = A.B^{-1}$ where B^{-1} represents the inverse of B. This result motivated us to search for the division of two non-zero, non-square matrices. For simplicity and clear understanding, we illustrate below the division of two non-zero, non-square matrices by employing the solutions of system of double Diophantine equations.

Illustration: 1

$$\text{Let } A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}$$

$$\text{Assume } \frac{C}{A} = B \tag{1}$$

Since, C is a matrix of order 2x2 and A is a matrix of type of order 2x3, observe that B should be a matrix of order 3x2.

$$\text{Consider } \begin{pmatrix} x & X \\ y & Y \\ z & Z \end{pmatrix} \tag{2}$$

Now, (1) is written as $C = AB$

$$\text{ie., } \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} x & X \\ y & Y \\ z & Z \end{pmatrix} = \begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix} \tag{3}$$

(3) is equivalent to the following equations

$2x + 3y + z = 14$ (4)

$5x + y + 4z = 25$ (5)

$2X + 3Y + Z = 14$ (6)

$5X + Y + 4Z = 21$ (7)

Considering (4) and (5), as a system of double Linear Diophantine equations and performing a few calculations, we can get the solution for x,y,and z. From (4) and (5)

2x+3y=14-z
5x+y=25-4z
8x+12y=56-4z
5x+y=25-4z
3x+11y=31 ---------- (8)

(8)is a diophantine equation, gcd(3,11)=1 and 1/31. So it has a solution.
Now consider,
3x+11y=31 ---------- 8
By inspection,
$X_0=14_1 y_0=-1$ is a solution.
The other solutions are given by
$x=x_0-(b/d)$ t
$y=y_0+(a/d)$ t ,, t∈I where a=3; b=11
∴ x = 14-11t
Y = -1+3t ,, t∈I

<u>To find z</u>

Now,(4)-(5) gives
-3x+2y-3z=-1
3z=-3x+2y+11
3z=-3(14-11t)+2(-1+3t)+11
∴z=13t-11
∴ Equations (4) and (5)are satisfied by
∴ x = 14-11t

y = -1+3t

z = -11+13t } t∈I ----------(9)

Now from(6) 2X+3Y=17-Z

From (7,) 5X+Y=21-4Z

8X+12Y=68-4Z ----------(10)

5X+Y=21-4Z ----------(11)

(9)-(10) gives

3X+11Y=47 ----------(12) This is a diophantine equation and gcd(3,11)=1 and 1/47. So it has a solution

By inspection, $x_0=23$ $y_0=-2$ is a solution

The other solutions are given by

$X=x_0-(b/d)\,T$ where a=3; b=11 , T∈I

$Y=y_0+(a/d)\,T$

∴ X=23-11 T

∴Y= -2+3 T T∈I

To find Z

(7)-(6) gives,

3X-2Y+3Z=4

∴ 3Z=4-3X+2Y

3Z=4-3(23-11T)+2(-2+3T)

Z=-23+13T, T∈I

So we have the solution for the elements of B as follows

$$\left.\begin{aligned} x &= 14-11t \\ y &= 3t-1 \\ z &= 13t-11 \end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned} X &= 23-11T \\ Y &= 3T-2 \\ Z &= 13T-23 \end{aligned}\right\} \qquad (14)$$

Substituting (13) and (14) in (2), note that

$$B=\begin{pmatrix} 14-11t & 23-11T \\ 3t-1 & 3T-2 \\ 13t-11 & 13T-23 \end{pmatrix}$$

Which is the required division of $\frac{C}{A}$

Verification : Eg 1

For t=1 and T=1

$$B=\begin{pmatrix} +3 & 12 \\ 2 & 1 \\ 2 & -10 \end{pmatrix}$$

Consider $A=\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}$ $C=\begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}$

Since C/A=B

We Show that C=AB

Consider $AB=\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}\begin{pmatrix} +3 & 12 \\ 2 & 1 \\ 2 & -10 \end{pmatrix}$

$$=\begin{pmatrix} 6+6+2 & 24+3-10 \\ 15+2+8 & 60+1-40 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}=C$$

Eg 2

$A=\begin{pmatrix} 2 & 3 & 1 \\ 5 & 4 & 1 \end{pmatrix}$ $C=\begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}$

For T=-1, T=2,

$$B=\begin{pmatrix} 25 & 1 \\ -4 & 4 \\ -24, & 3 \end{pmatrix}$$

$$AB=\begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}=C$$

Eg 3

For t=3, T=-1

$$B=\begin{pmatrix} -19 & 34 \\ 8 & -5 \\ 28 & -36 \end{pmatrix}$$

$$AB=\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}\begin{pmatrix} -19 & 34 \\ 8 & -5 \\ 28 & -36 \end{pmatrix}=\begin{pmatrix} -38+24+28 & 68-15-36 \\ -95+8+112 & 170-5-144 \end{pmatrix}$$

$$=\begin{pmatrix} 14 & 17 \\ 25 & 21 \end{pmatrix}$$

So C/A has infinitely many solutions.

Illustration 2

$$\text{Let } A=\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}, \quad C=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Assume $\frac{C}{A} = B$ (1)

Since, C is a matrix of order 2x1 and A is a matrix of order 2x3, observe that B should be a matrix of order 3x1.

Consider $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (2)

Now, (1) is written as $C = AB$

ie., $\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (3)

(3) is equivalent to the following equations
2x+3y+z =2
5x+y+4z=3
Or 8x+12y+4z =8
5x+y+4z =3

Subtracting,3x+11y =5, Which is a Diophantine Equation with gcd (3,11) =1
And 1 divides 5.So it has a solution. By inspection, x_0 =-2 , y_0 =1 is a particular solution
The other solutions are given by,
$x=x_0-(b/d)\ t$
$y=y_0+(a/d)\ t$,, t∈I where a=3; b=11
∴ x = -2-11t
Y = 1+3t t∈I

To find Z
(2)-(1) gives
3x-2y+3z =1,
3z=1-3x+2y =1-3(-2-11t)+2(1+3t)
Z =3 +13t t∈I

So the solution of (2) is given by

$B = \begin{pmatrix} -2-11t \\ 1+3t \\ 3+13t \end{pmatrix}$ t∈I

Eg.1. For t=1,x=-13, y=4,z=16

$B = \begin{pmatrix} -13 \\ 4 \\ 16 \end{pmatrix}$

and B satisfy C=AB or C/A =B

Eg 2. For t=-1,x =9,y =-2, z = -10

$B = \begin{pmatrix} 9 \\ -2 \\ -10 \end{pmatrix}$ Also it satisfy C=AB or C/A =B

where

$A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Conclusion:

In this paper, we have presented the division of two matrices in which one is a non-singular square matrix and the other one is a rectangular matrix. In conclusion, it is observed that one may find division of two non-zero, non-square matrices leading to infinitely many solutions for the division.

References

1.Elementary number theory by David M Burton
2.Diophontine Equations from Wolfram MathWorld..